JOURNAL OF ALGEBRA 18, 506–510 (1971)

Some Sylow 2-Groups Which Cannot Occur in Simple Groups

PAUL CHABOT

Department of Mathematics, University of Notre Dame

Communicated by Richard Brauer

Received February 11, 1970

1. INTRODUCTION

In this paper we will define a certain class **P** of 2-groups P and then prove the following theorem.

THEOREM 1.1. *Let $P \in \mathbf{P}$ be a Sylow 2-group of G. Then, $Z^*(G) > O_{2'}(G)$.*

The motivation for this result lies in the fact that, for q odd, the Sylow 2-groups of the symplectic groups $Sp(2n, q)$ are members of **P**.

The method is basically an involved application of the Krull–Schmidt theorem. For any $P \in \mathbf{P}$, we will demonstrate that there is a well-determined involution $\tau \in P$ such that $\langle \tau \rangle$ char. $C_P(x)$ for all involution $x \in P$. It is well-known that this implies that τ lies in $Z^*(G) = Z(G \bmod O_{2'}(G))$.

All groups considered are finite and unexplained notation may be found in Gorenstein [4].

2. DEFINITION OF P

Let **X** denote the class of all non-Abelian indecomposable 2-groups X with $\Omega_1(X) \leqslant Z(X)$ such that there is an involution τ_X in the commutator group X' with $\langle \tau_X \rangle$ characteristic in X. Let τ be a choice function on **X** such that for any $X \in \mathbf{X}$ we have that $\tau(X)$ is an involution of X' with $\langle \tau(X) \rangle$ characteristic in X.

DEFINITION. Let **P** be the smallest class of 2-groups subject to the restrictions:

(i) $\mathbf{X} \subseteq \mathbf{P}$.

(ii) If $X \in \mathbf{P}$, then $X \times A \in \mathbf{P}$, where A is any abelian 2-group.

(iii) If $X \in \mathbf{P}$, then $X \sim Z_2 \in \mathbf{P}$. (wreath product).

(iv) If $X, Y \in \mathbf{P}$, then $X \times Y \in \mathbf{P}$.

We now establish an important property of the class $\mathbf{P}$.

LEMMA 2.1. *If x is an involution of $P \in \mathbf{P}$, then $C_P(x) \in \mathbf{P}$.*

Proof. We proceed by induction since the statement is obvious for $P \in \mathbf{X}$ because then $\Omega_1(P) \leqslant Z(P)$ and $C_P(x) = P$ for all involutions $x \in P$. We consider our definition of the class $\mathbf{P}$ and quickly conclude that the lemma is trivially true for steps (ii) and (iv) by induction. Hence, we must only worry about step (iii).

Thus, let $P = (X \times X^u)\langle u\rangle$, $u^2 = 1$ and let x be any involution of P. If $C_P(x) \leqslant X \times X^u$, then $C_P(x) = C_{X-X^u}(x) \in P$ by induction. Thus, we may assume $C_P(x) \nleqslant X \times X^u$. If $x \in P - (X \times X^u)$, then we note that $X^x = X^u$ and, after a little thought, conclude that

$$C_P(x) = \langle x\rangle \times \{zz^x \mid z \in X\} \simeq \langle x\rangle \times X \in \mathbf{P}.$$

If $x \in X \times X^u$, let $x = x_1x_2$ and let $v_1v_2u \in C_P(x)$, where $x_1, v_1 \in X$ and $x_2, v_2 \in X^u$. So $x^{v_1v_2u} = x_2$ and $x^{v_1v_2u} = x_1$. Consider the conjugate $\bar{x} = x^{v_1}$ of x. Let $\bar{x} = \bar{x}_1\bar{x}_2$, where $\bar{x}_1 = x_1^{v_1} = x_1^{v_1v_2}$ and $\bar{x}_2 = x_2^{v_1} = x_2$. Thus, $\bar{x}_1{}^u = x_1^{v_1v_2u} = x_2 = \bar{x}_2$ and $\bar{x}_2{}^u = x_2{}^u = x_1^{v_1v_2} = \bar{x}_1$. Thus, $u \in C_P(\bar{x})$. Therefore, $C_P(x) \simeq C_P(\bar{x}) = (C_X(\bar{x}_1) \times C_{X^u}(\bar{x}_2))\langle u\rangle = C_X(\bar{x}_1) \sim Z_2 \in \mathbf{P}$ by induction, since $C_X(\bar{x}_1) \in \mathbf{P}$ by induction. This completes the proof of the lemma.

3. MAIN LEMMA

We extend our choice function τ to all of $\mathbf{P}$ by defining

(i) $\tau(X \times A) = \tau(X)$, A Abelian.

(ii) $\tau(X \sim Z_2) = \tau(X)\tau(X^u)$, where $X \sim Z_2 = (X \times X^u)\langle u\rangle$ and $u^2 = 1$.

(iii) $\tau(X \times Y) = \tau(X)\tau(Y)$.

LEMMA 3.1. *The involution $\tau(P)$ is well-defined for all $P \in \mathbf{P}$, and $\tau(C_P(x)) = \tau(P)$ for an involution $x \in P$.*

Proof. We proceed by double induction. Our first ordering is just inclusion on $\mathbf{P}$. For any $P \in \mathbf{P}$, our second ordering is on $\{C_P(x) : x$ involution of $P\}$ and is given by $|C_P(x)|$.

We begin our induction by simply observing that $\tau(P)$ is already well-defined for $P \in \mathbf{X}$ and that $C_P(x) = P$ for all involutions $x \in P \in \mathbf{X}$, since $\Omega_1(P) \leqslant Z(P)$.

Now, we observe that if $X \in \mathbf{P}$, then any non-Abelian indecomposable factor of X also lies in $\mathbf{P}$. Let $X = X_1 \times \cdots \times X_n \times B$ be a decomposition of $X \in P$ into non-Abelian indecomposable X_i and an Abelian group B. Let $P = X \times A$ for some Abelian $A \neq 1$. By induction, $\tau(X)$ is defined, and $\tau(X) = \tau(X_1) \cdots \tau(X_n)$. Clearly, $\tau(X) \in X' = X_1' \times \cdots \times X_n' = [X \times A, X \times A]$, since each $\tau(X_i)$ lies in X_i'. Now, let $\alpha \in \operatorname{aut}(X \times A)$. So we have

$$\begin{aligned} X_1 \times \cdots \times X_n \times B \times A = X \times A &= (X \times A)^\alpha \\ &= X_1^\alpha \times \cdots \times X_n^\alpha \times B^\alpha \times A^\alpha. \end{aligned}$$

By the Krull–Schmidt theorem, there is a one-to-one correspondence between the X_i and the X_i^α such that corresponding groups are centrally isomorphic. Thus, there is a central isomorphism $\varphi_i : X_i \to X_j^\alpha$, for some j. But, then, $\varphi_i(\tau(X_i)) = \tau(X_i)$, since $\tau(X_i) \in X_i'$. Now, $\langle \tau(X_i) \rangle$ char X_i implies that $\tau(X_i) = \tau(X_j^\alpha) = \tau(X_j)^\alpha$. Hence, α permutes the $\tau(X_j)$ and thus fixes $\tau(X)$. Therefore, the extension of τ is well-defined by step (i). It is clear that essentially the same argument shows that the extension of τ is well-defined by step (iii), because by induction we would have $\tau(X)$ and $\tau(Y)$ well-defined. The statement that $\tau(C_P(x)) = \tau(P)$ follows trivially by induction for steps (i) and (iii), since in these cases the centralizer may be written as a direct product of centralizers in smaller P's.

Thus, we are left to prove that the lemma is true in step (ii) of the extension of τ to $\mathbf{P}$. Let $P = (X \times X^u)\langle u \rangle$, $u^2 = 1$. Suppose we have a different presentation for $P = X \sim Z_2$; say, $P = (Y \times Y^v)\langle v \rangle$, $v^2 = 1$. We must show that $\tau(X)\tau(X^u) = \tau(Y)\tau(Y^v)$.

We first argue that $\tau(C_P(x)) = \tau(X)\tau(X^u)$ for any involution $x \in P - (X \times X^u)$. Let $C = C_P(x) \cap (X \times X^u)$. Then $C = \{zz^x \mid z \in X\}$ is isomorphic to X and $(X \times X^u) = CX^u$. Thus, $\tau(X) \equiv \tau(C) \bmod X^u$. Similarly, we have $\tau(X^u) \equiv \tau(C) \bmod X$. Thus, $\tau(C) = \tau(X)\tau(X^u)$. Now, we simply note that $C_P(x) = \langle x \rangle \times C$ and invoke induction to get $\tau(C_P(x)) = \tau(C) = \tau(X)\tau(X^u)$.

Similarly, $\tau(C_P(v)) = \tau(Y)\tau(Y^v)$. We now argue that $\tau(C_P(v)) = \tau(X)\tau(X^u)$. If $v \in P - (X \times X^u)$, this equality is clear from the previous paragraph. Hence, assume $v \in X \times X^u$. If $C_P(v) \leqslant X \times X^u$, then by induction we get $\tau(C_P(v)) = \tau(C_{X \times X^u}(v)) = \tau(X \times X^u) = \tau(X)\tau(X^u)$. Thus, we need now only treat the case that $C_P(v) \nleqslant X \times X^u$. Let $v = v_1 v_2$, where $v_1 \in X$ and $v_2 \in X^u$. Let $x_1 x_2 u \in C_P(v)$ where $x_1 \in X$ and $x_2 \in X^u$. Then, $v_1^{x_1 x_2 u} = v_2$ and $v_2^{x_1 x_2 u} = v_1$. Consider the conjugate $\bar{v} = v^{x_1}$ of v. Let $\bar{v}_1 \bar{v}_2 = \bar{v}$ where

$\bar{v}_1 = v_1^{x_1} = v_1^{x_1 x_2}$ and $\bar{v}_2 = v_2^{x_1} = v_2$. Thus, $\bar{v}_1{}^u = v_1^{x_1 x_2 u} = v_2 = \bar{v}_2$ and $\bar{v}_2{}^u = v_2{}^u = v_1^{x_1 x_2} = \bar{v}_1$. Thus, $u \in C_P(\bar{v})$. Therefore,

$$C_P(\bar{v}) = (C_X(\bar{v}_1) \times C_{X^u}(\bar{v}_2))\langle u \rangle.$$

By induction, $\tau(C_X(\bar{v}_1)) = \tau(X)$ and $\tau(C_{X^u}(\bar{v}_2)) = \tau(X^u)$ and thus, $\tau(C_P(\bar{v})) = \tau(X)\tau(X^u)$. Now we observe that $\tau(X)\tau(X^u) \in Z(P)$ and conjugate this last equality by x_1^{-1} to obtain $\tau(C_P(v)) = \tau(X)\tau(X^u)$. This completes the proof that τ is well defined.

Finally, let x be any involution of $P = (X \times X^u)\langle u \rangle$. We now use the previous argument with $v = x$ to prove that $\tau(C_P(x)) = \tau(P)$ and thereby complete the proof of the lemma.

4. Main Theorem

Using the theorem of Alperin [1] in conjunction with Glauberman's Z^*-theorem [3], it is a fairly easy exercise to prove the following well-known result.

Theorem 4.1. *Suppose τ is an involution of a Sylow 2-group P such that $\langle \tau \rangle$ char $C_P(x)$ for all involutions $x \in P$. Then, $\tau \in Z^*(G)$.*

In view of this theorem, our main theorem follows trivially as a corollary to Lemma 3.1. In fact, we have

Theorem 4.2. *Let τ be a choice function on $\mathbf{X}$ such that for any $X \in \mathbf{X}$, $\tau(X)$ is an involution of X' and $\langle \tau(X) \rangle$ char X. Then τ has an extension to $\mathbf{P}$ satisfying the following:*

Whenever $P \in \mathbf{P}$ is a Sylow 2-group of a group G, then $\tau(P) \in Z^*(G)$.

Finally, let $\mathbf{X}_0$ denote the set of generalized quaternions groups. Then, $\mathbf{X}_0 \subseteq \mathbf{X}$. Let $\mathbf{B}_0$ be the smallest class of 2-groups subject to $\mathbf{X}_0 \subseteq \mathbf{B}_0$, and if $X \in \mathbf{B}_0$, then $X \sim Z_2 \in \mathbf{B}_0$. Now, let $\mathbf{P}_0$ be the smallest class subject to $\mathbf{B}_0 \subseteq \mathbf{P}_0$, and if $X, Y \in \mathbf{P}_0$, then $X \times Y \in \mathbf{P}_0$.

We remark, see [2], that $\mathbf{P}_0$ represents exactly those 2-groups which appear as Sylow 2-groups of symplectic groups $Sp(2n, q)$, for q odd. It is obvious that $\mathbf{P}_0 \subseteq \mathbf{P}$ and that any one of our choice functions restricts to a unique choice function τ_0 on $\mathbf{P}_0$ which is the extension of the choice function on $\mathbf{X}_0$ that chooses the unique involution of X for each $X \in \mathbf{X}_0$. Thus, we have the following corollary.

COROLLARY 4.3. *Let $P_i \in \mathbf{B}_0$, $i = 1,\ldots, n$ and set $\tau_i = \tau(P_i)$, the unique involution of $Z(P_i)$. If $P = P_1 \times \cdots \times P_n$ is a Sylow 2-group of G, then $\tau = \tau_1 \cdots \tau_n \in Z^*(G)$.*

ACKNOWLEDGMENTS

I would like to thank Professor W. Wong for initially posing this question and also for pointing out errors in the original proof.

REFERENCES

1. J. L. ALPERIN, Sylow intersections and fusion, *J. Algebra* **6** (1967), 222–241.
2. R. CARTER AND P. FONG, The Sylow 2-subgroups of the finite classical groups, *J. Algebra* **1** (1964), 139–151.
3. G. GLAUBERMAN, Central elements in core-free groups, *J. Algebra* **4** (1966), 403–420.
4. D. GORENSTEIN, "Finite Groups," Harper and Row, New York, 1968.